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An Accurate Compact Treatment of Pressure for Colocated Variables

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We propose a new scheme for the pressure treatment in computations of incompressible flow using a colocated grid arrangement. To avoid oscillations associated with the sparse non-compact stencil, we introduce a compact fourth-order equivalent of this stencil and study its advantages over the classical second-order averaging procedure. © 1999 Academic Press

Key Words: compact differencing; incompressible Navier–Stokes equation; velocity-pressure decoupling.

1. INTRODUCTION

When computing incompressible Navier–Stokes equations on a non-staggered (colocated) grid, a Poisson equation is to be solved for the pressure (or pseudo pressure in the fractional step formalism). The numerical instability, often referred to as the "checkerboard problem" or "odd–even decoupling problem" then has to be addressed. This problem arises when second-order central difference approximations are implemented for both the pressure gradient operator in the momentum equation and the divergence operator in the continuity equation (or, with the cell centered finite volume formalism, if fluxes are obtained by central differencing), and when the discrete Poisson equation is defined, in a consistent and conservative manner, as the product of these two operators [1, 2]. This Poisson equation corresponds to a non-compact sparse stencil and produces an oscillatory pressure field. Various approaches have been used to overcome this difficulty. Van der Wijngaart [3] proposed to filter out the oscillations. To introduce a coupling term, Russel and Abdallah [4] increased the order of the divergence operator at the cost of an enlargement of the stencil for the discrete Laplace operator. The most common approach (here referred to as "compact averaged") involves the derivation of a non-conservative compact pressure Poisson equation [2, 5–7]. This modification is introduced differently depending on the way the pressure is treated in the time integration. It can be understood as a smoothing of the pressure field through

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an artificial dissipation. The energy conserving property of the scheme is destroyed in this process. The error thus introduced in the discrete continuity equation is proportional to the fourth-order derivatives of the pressure [8]. The idea of the present method is to introduce a fourth-order "compact equivalent" to the conservative discrete pressure equation.

2. GENERAL METHODOLOGY

In the following, we will present the method using the fractional step formalism (first-order accurate in time [9]), though it can probably be generalized to the pressure correction approach or other schemes. We will thus suppose a velocity field $\boldsymbol{u}^*$, which does not satisfy the continuity equation (for instance obtained by time-advancing the Navier–Stokes equations without invoking continuity) and we want to project it onto a divergence-free field by subtracting the gradient of a pressure-like variable ϕ so that

$$\boldsymbol{u} = \boldsymbol{u}^* - \boldsymbol{\nabla}_h \phi. \tag{1}$$

Taking the divergence of this equation and requiring $\boldsymbol{u}$ to satisfy the continuity equation gives

$$\Delta_{2h}\phi = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*, \tag{2}$$

where Δ_{2h} stands for the second-order centered approximation of the Laplacian skipping the neighboring points (as represented on Fig. 1a). The sparse nature of this operator leads to pressure oscillations. A possible remedy to this problem is to interpolate variables linearly, which leads to a second-order approximation of (2)

$$\Delta_h \phi = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*. \tag{3}$$

We will refer to this in the sequel as the "compact averaged" scheme. This scheme can be interpreted as adding a second-order dissipative term to the pressure in order to damp the oscillations. In the following we will derive fourth-order compact equivalents of (2).

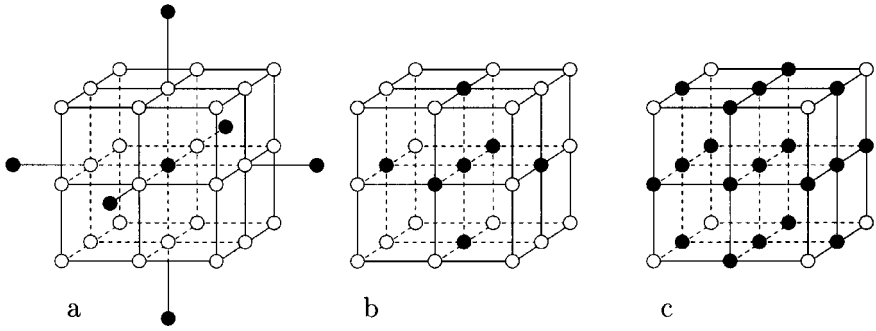

FIG. 1. Computational molecules for the three-dimensional pressure equation; only the black points are used for the computation. (a) The Δ_{2h} operator, being defined as the product of numerical divergence of the gradient, accurate but oscillating. (b) The 7 points compact operator obtained by interpolation. (c) The 19 points fourth-order compact equivalent operator.

2.1. One Dimension Compact Derivation

Using the expression of truncation error for the second-order finite difference scheme

$$\Delta_h \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 \phi}{\partial x^4} + \mathcal{O}(h^4), \tag{4}$$

one easily derives the second-order error introduced in replacing (2) with (3):

$$\Delta_h \phi = \Delta_{2h}\phi - \frac{h^2}{4}\frac{\partial^4 \phi}{\partial x^4} + \mathcal{O}(h^4). \tag{5}$$

The error thus introduced is of the same order as the one for the discretization scheme (see [2, 4]). However, the discrete Laplacian is only a second-order estimate of the discrete divergence of the gradient. Using the fractional step procedure described above (see [10, 11, 9]) thus leads to a modified velocity field in which numerical errors in the divergence are significantly higher than roundoff error.

The consistent derivation of the pressure-like equation gives

$$\Delta_{2h}\phi = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*, \tag{6}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{h^2}{3}\frac{\partial^4 \phi}{\partial x^4} = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^* + \mathcal{O}(h^4). \tag{7}$$

Using a Hermitian compact expression one can derive a fourth-order approximation of Eq. (7),

$$\Delta_h \phi + \frac{h^2}{4}\frac{\partial^4 \phi}{\partial x^4} = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^* + \mathcal{O}(h^4), \tag{8}$$

$$\Delta_h \phi = \left[I - \frac{h^2}{4}\Delta_h\right](\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*) + \mathcal{O}(h^4), \tag{9}$$

where I stand for the identity operator.

Note that this compact scheme was not derived to increase the accuracy of the continuous operator approximation as is the case for other compact formulations [12–18]. Instead it is derived to approximate to a higher accuracy the conservative non-compact stencil. The compact scheme we propose (9) is still second-order accurate, but has the same second-order behavior as the conservative sparse stencil (6). The overall accuracy of the discretization is thus second-order in space and first-order in time, but the dilatation effects associated with the non-conservative treatment of the pressure are reduced to a fourth order. Note that this approach also shares some similarity with what is refered to as "improving the order of approximation" in the Support Operator formalism [19].

2.2. Three-Dimensions Generalization

The idea of the previous paragraph can be adapted to three-dimensional problems, though this is not straightforward. The original non-compact formulation can be written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{h_x^2}{3}\frac{\partial^4 \phi}{\partial x^4} + \mathcal{O}\left(h_x^4\right) + \frac{\partial^2 \phi}{\partial y^2} + \frac{h_y^2}{3}\frac{\partial^4 \phi}{\partial y^4} + \mathcal{O}\left(h_y^4\right) + \frac{\partial^2 \phi}{\partial z^2} + \frac{h_z^2}{3}\frac{\partial^4 \phi}{\partial z^4} + \mathcal{O}\left(h_z^4\right) = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*. \tag{10}$$

Note that the second-order terms cannot be easily expressed in terms of $\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*$.

2.2.1. *Single Step Procedure*

Following the method used by Spotz and Carey [15] (but with a non-uniform grid) a compact fourth-order approximation to (10) can be derived in the form

$$\Delta_x\phi + \Delta_y\phi + \Delta_z\phi - \frac{1}{4}\Delta_x\left[h_y^2\Delta_y\phi + h_z^2\Delta_z\phi\right] - \frac{1}{4}\Delta_y\left[h_x^2\Delta_x\phi + h_z^2\Delta_z\phi\right]$$

$$-\frac{1}{4}\Delta_z\left[h_x^2\Delta_x\phi + h_y^2\Delta_y\phi\right] \simeq \left[I - \frac{h_x^2}{4}\Delta_x - \frac{h_y^2}{4}\Delta_y - \frac{h_z^2}{4}\Delta_z\right](\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*), \quad (11)$$

where Δ_x, Δ_y, Δ_z stand for the h-discrete three point Laplace scheme respectively in the x, y, and z directions. Notice that this expression not only involves a modification of the right-hand side (rhs), but also a modification of the operator on the left-hand side (lhs), with the addition of extra diagonal terms. The corresponding 19 points *computational molecule* is displayed in Fig. 1c.

2.2.2. *Two Step Procedure*

The previous approach corresponds to a direct adaptation of the one-dimensional idea to the three-dimensional case. It achieves fourth-order accuracy, but only at the cost of a modification of the lhs itself for the treatment of cross derivatives. In two spatial dimensions this would only increase the stencil from five to nine points. In three space dimensions however this increase is much more important as the standard seven-points stencil has to be modified to a nineteen-points molecule. This significantly increases the computational time required. It was found (see Subsection 3.1) that the number of iterations to resolve this problem with a CGSTAB algorithm [20] is about three times that required by the nine-points scheme.

We propose here an alternative two-step approach that allows fourth-order accuracy at twice the computational cost of the second-order interpolated scheme. Equation (10) can be approximated with fourth-order accuracy using a two-step procedure. The first step is a second order approximation to the second-order truncation terms written as

$$\Delta_h\psi = \left(h_x^2\frac{\partial^4}{\partial x^4} + h_y^2\frac{\partial^4}{\partial y^4} + h_z^2\frac{\partial^4}{\partial z^4}\right)\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*. \quad (12)$$

Technically, evaluation of the rhs of (12) involves higher order derivatives of $\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*$ which implicates larger stencils (five-point stencils in 1D).

A second step uses ψ as a correction term:

$$\Delta_h\phi = \boldsymbol{\nabla}_h \cdot \boldsymbol{u}^* - \frac{1}{4}\psi. \quad (13)$$

Note that ψ scales as $\mathcal{O}(h^2)$ and that (13) is thus consistent with (2).

The pressure thus defined satisfies (2) to the fourth-order (as with the previous method) and the computational cost is exactly twice the cost of the second order scheme as each of the steps requires the resolution of a seven-points compact Laplace operator (of the form displayed in Fig. 1b).

Though the lhs stencils are simple and compact, the rhs of the first step involves a non-compact stencil. However, this does not increase the numerical cost as this term does not need to be inverted.

3. NUMERICAL EXAMPLES

3.1. *One-Dimensional Function*

Though incompressible flow in one dimension of space is a rather limited notion, we want to test the ability of the scheme on a simple problem. Let us assume a velocity field $\boldsymbol{u}^*$ of the form

$$\boldsymbol{u}^* = \sin(4\pi x), \qquad x \in [0, 1]. \tag{14}$$

Application of the procedure described in Subsection 2.1 to this one-dimensional function should obviously lead to

$$\boldsymbol{u} = \boldsymbol{u}^* - \boldsymbol{\nabla}_h \phi \equiv \boldsymbol{0}. \tag{15}$$

The compact fourth-order scheme used here (9) is defined implicitly as

$$\frac{1}{h^2}(\phi_{i-1} - 2\phi_i + \phi_{i+1}) = \frac{1}{4}(\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*)_{i-1} + \frac{1}{2}(\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*)_i + \frac{1}{4}(\boldsymbol{\nabla}_h \cdot \boldsymbol{u}^*)_{i+1}. \tag{16}$$

We report in Table I the errors (defined as the maximum of the absolute value of $\boldsymbol{\nabla}_h \cdot \boldsymbol{u}$) with varying discretizations, solutions are represented in Fig. 2. The orders of the various schemes are vindicated.

It should be noted that, in one dimension, the compact scheme only requires an additional multiplication by a tridiagonal matrix on the rhs (as compared with the compact averaged scheme). The computational cost of the compact equivalent scheme is thus very similar to that of the compact averaged approach.

3.2. *Three-Dimensional Flow*

We have constructed a code for the purpose of studying three-dimensional rapidly rotating magnetohydrodynamic buoyancy-driven turbulence. It is known [24] that buoyancy-affected flows require a fine pressure-velocity coupling. Furthermore, in the physical problem that motivated this work, we expect from previous studies [21] plate-like shear zones. It is important to compute gradients in these regions accurately. This motivated the choice of

TABLE I
Errors (Maximum of the Absolute Value) in the Numerical Divergence with Varying Grid Sizes (N)

N	C.A.	C.E.
20	1.12	0.11
40	0.30	$7.4 \cdot 10^{-3}$
80	$7.70 \cdot 10^{-2}$	$4.74 \cdot 10^{-4}$
160	$1.93 \cdot 10^{-2}$	$2.98 \cdot 10^{-5}$
320	$4.84 \cdot 10^{-3}$	$1.86 \cdot 10^{-6}$

Note. As expected, the compact averaged scheme's error (C.A.) evolves as $1/N^2$, while the fourth-order compact equivalent scheme's error (C.E.) evolves as $1/N^4$. The solution derived using the sparse non-compact scheme is of the order of the numerical zero (about 10^{-13}).

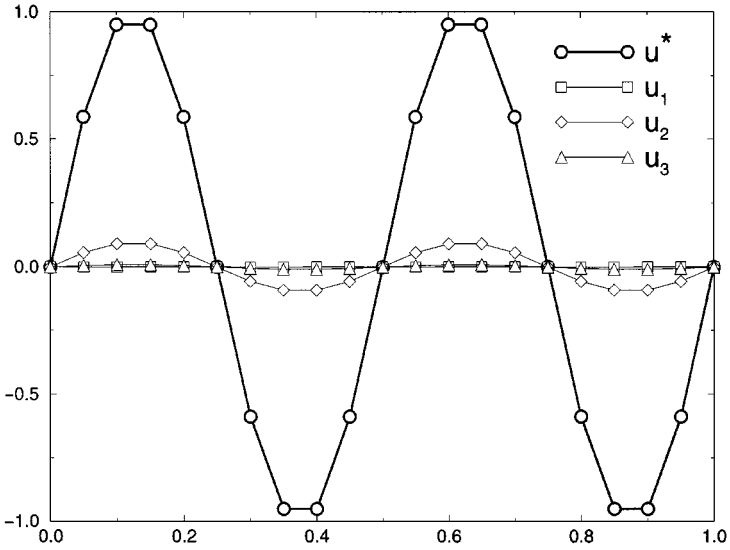

FIG. 2. Starting with a velocity field $\boldsymbol{u}^*$, which does not satisfy the continuity condition, we want to compute the velocity field $\boldsymbol{u}$ by subtracting the gradient of a pseudo-pressure. This is done by solving a discrete Poisson equation, using for $\boldsymbol{u}_1$ the natural expression Δ_{2h} (accurate but leading to oscillations); for $\boldsymbol{u}_2$ the compact averaged scheme Δ_h; for $\boldsymbol{u}_3$ the fourth-order compact equivalent of Δ_{2h}. Errors when varying the discretization are reported in Table I.

colocated variables though the geometry is rather simple. In a work in preparation, we test adaptations of the ENO scheme [22, 23] for advective transport on incompressible flows. This also motivated the development of the present equivalent scheme for the pressure gradient.

We use here our code with no magnetic field and no rotation and we study a simple laminar buoyancy-driven flow of a Boussinesq fluid in a fully periodic domain. Time integration is performed using the optimal second-order TVD (total variation diminishing) Runge–Kutta method [23] (see also [11]). We report here some results obtained with the above described scheme.

Figure 3 displays the time evolution of the divergence after each full time step. The single step compact equivalent approach is found to give accurate results in three dimensions. The single step procedure leads with a $50 \times 50 \times 25$ grid to a decrease of the error in the numerical divergence of a coefficient about 14. This method led to a significant increase of the required CPU time (about a factor three with the compact averaged scheme). The two-step algorithm is found to give slightly better and more regular results (probably because we achieve a better resolution of the seven-point stencil with our iterative solver). As expected it requires twice the computational time of the compact averaged method, although it was found numerically that ψ does not need to be computed with as great an accuracy as ϕ. Relaxing the precision constraint on ψ we were able to obtain a fourth-order accurate solution with only 50% more time than the second-order averaged computation.

4. CONCLUSION

We introduced a compact fourth-order equivalent of the pressure equation for the discrete resolution of the incompressible Navier–Stokes equation. This approach suppresses the

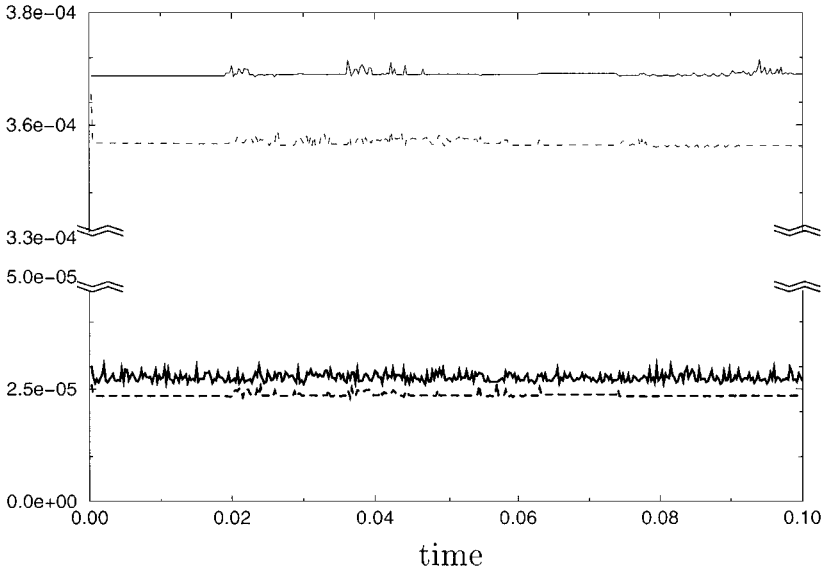

FIG. 3. Time evolution of the numerical divergence in three-dimensional simulations of a buoyancy driven convective flow (resolution is $50 \times 50 \times 25$). The computation starts with a flow computed using the C.A. method. Integration with the C.A. method gives the solid line curve. Introduction of the mass matrix of the single step fourth-order compact equivalent scheme improves the solution a little, though the overall accuracy is still second order (dashed curve). Modification of the operator to obtain fourth-order accuracy requires a stronger computational effort but yields much better results (bold curve). Finally the two-step method (dashed bold) gives as good (if not better) results for a lower computational effort. None of these simulations is oscillating because of the compact nature of the operators used.

spatial odd–even decoupling of the pressure field without adding a second-order damping term (as was previously the case). This significantly reduces the residual errors in the discrete continuity equation.

Three-dimensional simulations have been performed and this approach has been shown to give satisfactory results at a reasonable computational cost (using a two-step algorithm) for the fully periodic buoyancy driven flow that motivated this study.

Further important issues remain to be addressed about the scheme introduced here, such as the treatment of boundary conditions (e.g., see [25]) as well as the generalization of this technique to unstructured grids.

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